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Abstract

Diagrammatic, analogical or iconic representations are often contrasted with linguistic or logical representations, in which the shape of the symbols is arbitrary. Although commonly used, diagrams have long suffered from their reputation as mere tools, as mere support for intuition. We list here the main characteristics of diagrammatic inferential systems, and defend the idea that heterogeneous representation systems, including both linguistic and diagrammatic representations, offer real computational perspectives in knowledge modeling and reasoning.

Introduction

With the advent of printing, textual transmission became dominant in communication. Today, partly because of the development of technologies to create, reproduce and transmit images, figures are again in the spotlight. But their increased use in communication is also due to the relatively recent discovery that graphical representations also can convey abstract meanings, by means of visual metaphors. Indeed the use of space makes it possible to capitalize on the considerable human abilities to make spatial inferences.

Although commonly used in Science (as in Physics, Mathematics and Logic) diagrammatic representations have long suffered from their reputation as mere tools in the search for solutions, as mere support for intuition. This general prejudice against diagrams has been strongly denounced in the 90's by Barwise and Etchemendy, who defended the idea that a general theory of valid inferences can be developed independently of the modes of representations (as implicit in Peirce). In fact, diagrams can also be considered as structured syntactic objects, which can be used for correct reasoning in a formal perspective. Their work yields to the rigorous demonstration, done by Sun-Joo Shin in her PhD thesis, that diagrammatic systems can be formally proved as being sound and complete[1] (Shin, 1994).

Historically, the use of circles to illustrate relations between classes has often been attributed to Euler (1768), but he was in fact preceded by others (as by Leibniz), one century earlier. But these circles were unable to express all the relations between two classes on the same diagram, and this reduced considerably their expressive and deductive power. In the nineteenth century, Venn found a way of correcting this lack. Twenty years later, Peirce added to his system the possibility to express existential assertions (marked by a cross) and disjunctions (by means of lines joining crosses and small circles) – small circles being used to indicate the negation of existence in empty areas, instead of hatching as in Venn diagrams (Peirce, 1933)

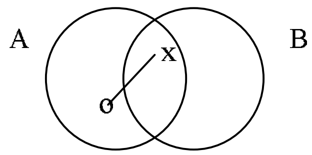

Figure 1: All A are B and some A are B (Peirce)

The idea of transformation rules on diagrams is also due to Peirce. But the question of the validity of these rules required a supplementary proof, which S. Shin provided. To do so, Shin provided graphical systems with a syntax (defining the primitive graphical objects and the well-formed diagrams) and a semantics (here representing sets of objects with areas) as in modern logic systems. Her two systems, based on Venn-Peirce diagrams, allow the resolution of syllogisms constituted by assertions on sets by means of transformation rules on diagrams (using the notion of equivalence between diagrams, and allowing the derivation of one diagram from others, with deletion and unification graphic rules).

Diagrammatic systems properties

We list here the main characteristics of diagrammatic systems discussed in the literature. Note that these are neither necessary nor sufficient. For details, see (Recanati, 2005).

A structural homomorphism

Diagrammatic and linguistic representation systems may have several degrees of homomorphism between their representations and the situation they represent. Barwise and Etchemendy have emphasized that the main properties of diagrammatic systems are derived from the existence of a syntactical homomorphism such that, usually:

1. Objects are typically represented by icon tokens. It is often the case that each object is represented by a unique icon token and distinct tokens represent distinct objects.
2. There is a mapping ϕ from icon types to properties of objects.
3. The mapping ϕ preserves structure. For example, one would expect that:

[1] The only preceding attempt in this direction is due to J. Sowa in 1984 (Sowa, 1984).

a) If one icon type is a subtype of another (as in the case of shaded squares and squares, for example), then there is a corresponding subproperty relation among properties they represent.
b) If two icon types are incompatible (say squares and circles), then the properties they represent should be incompatible.
c) The converses of (a) and (b) frequently hold as well.

4. Certain relations among objects are represented by relations among icon tokens, with the same kinds of conditions as in (3a)-(3c).
5. Higher-order properties of relations among objects (like transitivity, reflexivity and the like) are reflected by the same properties of relations among icon tokens.
6. Every possibility (involving represented objects, properties, and relations) is representable. That is, there are no possible situations that are represented as impossible.
7. Every representation indicates a genuine possibility. (Barwise & Etchemendy, 1995)

But in many cases, diagrammatic system properties seem to be derived from the property of closure under constraints.

Closure under constraints

This is the most paradigmatic property of diagrammatic systems (first isolated by B&E). In such systems, the representations exhibit most of the previously cited characteristics. We reformulate this property as follows:

A diagrammatic system of representation is *closed under constraints* if, and only if, there exists a syntactical homomorphism requiring that all logical consequences of the represented situation be explicit in the representation.

Suppose you have to seat three people, Loana, Thierry and Claire, on aligned chairs. Knowing that Thierry is on the left of Loana, and Loana on the left of Claire, you can represent this situation diagrammatically by using three symbolic letters that you will place in accordance with these two hypotheses as in:

T L C

Conversely, in a linguistic representation system, you will write:

T on-the-left-of L
L on the-left-of C

The consequence of these two hypotheses, namely, that T is on the left of C, is explicit in the diagrammatic case, while it requires an inference mechanism (and a rule of transitivity) to be derived in the linguistic one.

Duality of linguistic and diagrammatic reasoning The preceding example illustrates the deep duality between linguistic and diagrammatic modes of reasoning. Linguistic (or traditional logical) reasoning requires (1) the representation of initial facts, (2) an explicit representation of abstract properties (or relations between dimensions) of the objects, and (3) a computational mechanism linking the two sources of information to establish the validity of a non-explicit consequence. To the opposite, diagrammatic reasoning (in closed under constraints systems) do not require the explicit representation of abstract properties, because *these properties are automatically taken into account by syntactic constraints on the representation itself.* They merely need the representation of facts, and to establish the validity of a consequence, the representations have only to be inspected to check whether the new fact is or not represented there. This makes these systems computationally very efficient[2].

Easy treatment of conjunction As just mentioned, the facts are then simply added within a (global) representation and their consequences automatically follow.

Difficulties with disjunction Diagrammatic representations lead frequently to consideration of alternatives, because self-contradictory situations cannot be represented on the same diagram[3].

Contradictions cannot be represented and each representation corresponds to a genuine situation To illustrate the last points let's take an example from B&E: « You are to seat four people, A, B, C and D in a row of five chairs. A and C are to flank the empty chair. C must be closer to the center than D, who is to sit next to B. From this information, show that the empty chair is not in the middle or on either end. Can you tell who is to be seated on the two ends? » (Barwise et Etchemendy, 1990).

A diagrammatic representation of a situation may consist in aligned letters (as in our first example), a cross to indicate the empty chair, and dashes for non-attributed chairs. Then, the first hypothesis requires distinguishing between three disjunctive cases:

1. A x C – –   2. – A x C –   3. – – A x C

The second hypothesis eliminates case 3, and the third (D next to B) case 2. (These cases are suppressed because B or D cannot be added to the representation).Then, case 1 yields to the two final possibilities :

1.1. A x C B D   1.2. A x C D B

Specificity and limited abstraction

Keith Stenning and Jon Oberlander (1995) have introduced three classes of representational systems: the MARS (Minimal Abstraction Representational Systems), the LARS (Limited Abstraction Representational Systems) and the UARS (Unlimited Abstraction Representational Systems).

[2] Insofar as the inspection procedures are not too costly.

[3] We have seen that Peirce diagrams enabled the representation of disjunctive cases on the same diagram. Nevertheless, the efficiency may be partly lost with disjunction or second order relation (as with implication). This is the case with the Venn-II diagrams system developed by Shin, which supports disjunctions but requires more complex diagrams. In this system, the simple inspection of a representation to check a property is not always possible, and it may require extra (graphical) computation, as in the linguistic case.

They argue that this hierarchy of representational systems is analogous to that of languages isolated by Chomsky, and they claim that most diagrammatic representation systems are LARS.

MARS, LARS and UARS A MARS is a system in which a representation corresponds to a unique model of the world under the considered interpretation. For instance, a table of 0 and 1, representing the values of unary predicates of objects in a world W, will be a minimal abstraction representation, if each cell of the table is occupied by 0 or 1, and nothing else. MARS have no abstraction capacity and each new dimension brings as much alternative worlds as the number of possible values in this dimension.

But you can augment the number of models captured by a MARS by introducing new symbols that allow abstracting on a certain dimension of the representation. For instance, in the preceding table, you can allow empty cells. Such systems can quantify massively on possible models, but cannot specify arbitrarily complex dependences between the specified dimensions. This is why S&O called them LARS. The abstraction is performed over models that differ on the values of objects among their dimensions, but not on the nature of relations or constraints that may exist between these dimensions. Only linguistic symbols, added to a representation, could allow the description of arbitrarily fine dependences between dimensions. Thus, for S&O, a system is a UARS, if it expresses dependences, inside a representation, with equations or others, or outside a representation, with linguistic assertions on the «keys» defining the representation itself. *Diagrammatic representations thus differ from linguistic ones by a limited abstraction power, which augment their computational efficiency.*

Determined character and specificity S&O identify the restricted capacity of diagrammatic systems with a property called *specificity*, which requires information of a certain kind to be specified in all interpretable representation. Perry and Macken have opposed to this too strong notion (= the mandatory specification of values of properties other than the one you try to represent) the notion of determined character (in the sense of Berkeley) – only what is really necessary for diagrammatic representations (Perry & Macken, 1996).

Berkeley's notion of a determined character is derived from the fact that it is not possible to represent an object as having a certain property, without representing at the same time a specified value for this property. Thus, I cannot represent a triangle on a mathematical figure, without ending with a particular triangle. As well, it is not possible to represent a colored object on a drawing without specifying its color, but I can perfectly say, « this object has an interesting color », without specifying which one[4]. But for P&M, the closure under constraints of B&E does not require the specificity of S&O, but another property, which they call *localization*.

Localization and perceptual inferences

J. Larkin and H. Simon have identified three advantages of diagrams on verbal descriptions for solving problems:

- Diagrams can group together all information that is used together, thus avoiding large amounts of search for the elements needed to make a problem-solving inference.
- Diagrams typically use location to group information about a single element, avoiding the need to match symbolic labels.
- Diagrams automatically support a large number of perceptual inferences, which are extremely easy for humans. (Larkin & Simon, 1987)

The first two properties are not clearly distinguished in the second point of L&S. The first is about what we call logical localization, and (2) refers both to logical localization and to what seems more distinctive of diagrammatic systems, and that we prefer to call spatial localization.

(Logical) localization (or *unique token constraint*) is the property of using only one token of a symbol to represent an object. This property disappears generally when you use a typed system. The omnipresence of representation of the same type designating the same object is thus observed in human language, where references to an object can be spread out everywhere in a document, so that information is not « localized » (Perry et Macken, 1996)**.** For P&M, this additional character is the one required to give diagrammatic systems the closure under constraints property, when combined with iconicity and a constraint and systematic homomorphism.

Spatial localization consists in using « places » or « loci» of a geometrically structured space (usually of dimension 2), to encode several features (as color, texture, forms, etc, are in images). The abundance of visual interferences of these encoded dimensions allows defining relations between crossing dimensions, which can be detected for free by our perceptual abilities[5].

Reducing the number of dimensions The encoding evoked here may lead to a reduction in the number of dimensions, and similar techniques (using maps or charts) are exploited today for this purpose in data mining.

Symmetrical arguments The frequency of symmetrical arguments in diagrammatic reasoning, which has been noted in the literature, may obviously come from spatial localization (for instance, there was an implicit symmetrical argument in our chairs problem). But we believe this is not a

[4] The analogical/digital distinction is also based for Dretske on a notion of specificity (Dretske, 1981, p.137). For him, every signal transmitting information necessarily carries this information under two aspects: an analogical form and a digital form. The analogical form always contains an additional specificity relative to the information properly conveyed by the digital form.

[5] The logician Jean Nicod, who developed cognitive models of euclidian spaces based on similarity relations, has analyzed these cognitive mechanisms with great subtlety.

specific feature of diagrammatic reasoning, since these arguments also appear in linguistic cases.

Iconicity

For Macken, Perry, and Cathy Hass (Macken et alii, 1993), this fundamental property allows representations with *Richly Grounded Meaning* (RGM). A RGM is a meaning whose relation to form is not arbitrary. This may come from several factors binding the form of the symbol to its meaning. An iconic sign may have a *Readily Inferable Meaning* (RIM), an *Easily Remembered Meaning* (ERM), or an *Internally Modifiable Meaning* (IMM). Road signs provide numerous examples of ERM and RIM (for instance, signposting bends). There also are in musical scores many examples of symbols having a RIM coming from their analogical character (as a crescendo situated under the stave).

Iconicity has been only very partially analyzed until now. We are convinced that it must be related to syntactical homomorphism because we believe that the main distinction between linguistic (or symbolic) representation systems and analogical representation systems (as diagrammatic systems) must be characterized in terms of the power of the meta-language required to provide the semantics of the system. In the analogical case, the meta-language requires reference to syntactical properties of the object language, while in the symbolic case, this is not obligatory. Let's take the example of Thierry, Claire and Loana, who are represented as « ordered » in the diagrammatic case. A minimal difference, but an essential one, between the two types of representations is the following:

(I) left-of (a, b) & left-of (b, c)

and (II) ordered ([a, b, c]) (or just [a, b, c])

There is an additional syntactical complexity for (II) which prevents its meaning, contrary to that of (I), from being described as a function of one argument of its predicate's meaning. Indeed, you can easily assign a meaning to the semantic equation:

$$[\![\text{left-of} (\square, \square)]\!] = [\![\text{left-of}]\!] ([\![\square]\!], [\![\square]\!])$$

while you cannot write anything else but:

$$[\![\text{ordered} ([a, b, c])]\!] = [\![\text{ordered}]\!] ([\![[a, b, c]]\!]),$$

which implies giving meaning, *at the meta-language level*, to a *configuration* of terms (the list figuring between simple square brackets). Therefore, the semantic descriptive meta-language must *offer possibilities of syntactical structuring of data similar to the ones figuring in the representation language*, because it will sometimes be necessary to assign them a meaning. This is not to say that all syntactical nuances of the representational system must be reflected in the interpretation system, because not all iconic representation features are interpreted in a diagrammatic representation. As well, syntactical structuring is not always necessary for an iconic feature of a symbol to be exploited (as with the use of bold to indicate the focus on one element).

Nevertheless, this shows that semantic compositionality relies on syntactic considerations, which leads to this interesting question: which syntax do we need (at the meta-language level) to describe human language?

Heterogeneous systems

The preceding section underlined the very interesting properties of diagrammatic inferential systems, but the request of a unique syntactical homomorphism is obviously restricting. Nevertheless, this shortcoming is eliminated from heterogeneous representation systems (HRS), which have the attractive feature of being able to benefit of opposite properties of logical and analogical systems.

Computational perspectives

Against all expectations, HRS may lead to amazing results in the domain of computational complexity. Diagrammatic systems have the fantastic property of bypassing computation, and the possibility of switching from one system to another opens up new research perspectives. The paradox is that a given demonstration may be limited by a minimal cost in *any* two systems, and still be *less costly* in a hybrid system binding the two. There is nothing sophistic here, because a heterogeneous system doesn't need a global language to bind its subsystems.

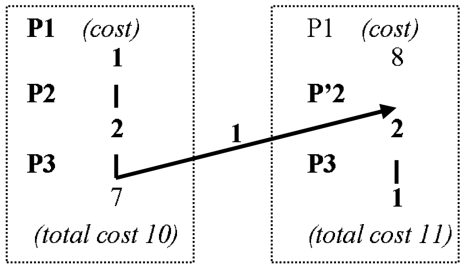

Figure 2: Minimal cost (*total 7*) of a diagonal proof

No doubt the supporters of traditional logic are ready to concede that deductive properties can be extended to non-linguistic representations in an HRS, but they may still be convinced that the final articulation of these subsystems requires a traditional intermediate language, in which any operation would have a cost, to be correctly based. But the articulation of different subsystems of representation in a complex representational system may be based simply on *the fact that they denote the same objects in the world.* Thus, two subsystems may denote different properties of objects, and what can be expressed in one subsystem need not be expressed in another; some information may nevertheless be transferred from one system to another, on the basis of safe correspondences, thus endowing the global system with superior inferential capacities. To underline this attractive fact, B&E gave a theoretical justification of two main algorithms implemented in *Hyperproof*[6] based on

[6] Hyperproof is an interactive software, designed to teach logic; a world of simple geometrical forms of various sizes, represented on the square of a checkerboard, is depicted using two formats: a logic language, and some visual representations on the checkerboard. The user can check the validity of new sentences, given these descriptions in both formats, by making « hybrid » proofs.

purely mathematical grounds, without using any intermediate language, lacking from their hybrid system.

Cognitive modeling

Logical formalisms are in many respects unsuited for the description of human inferential capacities. On top of their computational properties, HRS seem better suited for this task. This is the case for temporal and causal relationships, and no doubt in general, with respect to the main categories and relations we used to combine our mental representations and to situate ourselves in the surrounding world. This thesis is reinforced for time and space by the abundance of schemas found in most contemporary semantic works. Even though this fact is not a proof of the existence of diagrams in our mind, if they can explain the successes and failures of human inferences, they are of great significance.

Hybrid reasoning at MasterMind

We would like to conclude this talk by an illustrative example. The game of MasterMind[7] is well suited to the study of human reasoning, because it constrains the player to logical reasoning, not such a frequent occasion in everyday life. In (Recanati, 2004) we highlighted the hybrid character of the reasoning carried out in this game, the geometry of the grid supporting the pawns encouraging the players to develop diagrammatic representations. We have insufficient room here to report our observations, but we will give the flavor of them.

For most beginners, the reasoning is fragmented and opportunistic, and consists in partial deductions using several types of representations. In fact, most of the deductions are performed graphically while the model under construction is frequently described linguistically. Each attempt played on a row generates new interpretation schemas and new information, but these cannot always be combined with the preceding. Nevertheless, mental projections of new assumptions are regulary propagated from one row to another, to check whether or not the considered hypotheses are contradictory.

Experienced players develop properly graphical strategies of resolution, and project on the grid several types of geometrical representations. The use of graphic representations mitigates limitations in the cognitive capacities of the player, anchoring reasoning on inexpensive visual capacities, thus relieving verbal memory. But in return, the visual capacities being also restricted, the form of the diagrams used and the ordering of hypotheses are biased. For instance, the left-to-right order of pins and pawns, and the ease of visual translations, influence the choice of hypotheses to be considered first (as shown on Figure 3).

The grid ensures the memorizing of preceding attempts, but is also used as geometrical support for organizing proofs and backtracking. The reasoning begins with partial facts discovered on some attempt, but ends up being based on an interpretation of the first row. One frequently notes an ordering of global reasoning based on the vertical order of the rows, which helps the player in backtracking.

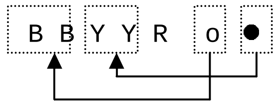

Figure 3: A favourite interpretation schema

The strategy of experienced players is based on the progressive construction of a model (very similar to the “mental models” of Johnson-Laird) by covering a tree of self-inclusive ordered models. They usually separate the play in two parts: determining first the colors, and then determining their places. In both phases of the game, they use representations (specific arrangement of pawns), which can be qualified as mental models. The interesting fact is that these models, which also correspond to the LARS of S&O, are constructed both *by increasing order of specificity and decreasing order of probability of appearance*. This makes backtracking easier and allows to converge quickly towards the solution. To illustrate this in a prototypical example, let's analyse the first four rows of Figure 4:

4.	R G R Y G	o o o ● ●
3.	R R R G G	o o o ●
2.	O O B B B	
1.	B B Y Y R	o ●

Figure 4: The grid at the middle of a game

The player begins on row 1 with a configuration, which is revealed to be statistically more informative than others. Given the answer on the right side, he considers first the interpretation of Figure 3, i.e. that one blue is correctly placed, one yellow misplaced, and that there is no red. (He might take in his hand a blue and a yellow pawn to help memorize, and note mentally that the three colors are exhausted). We will note this “mental model” by [1B] [1Y] (and no red) – using square brackets for the notion of exhaustivity defined by Johnson-Laird (1983). Then, he plays the second row, trying new places for blue (in anticipation of a future reasoning on places) and introducing a new color (orange). By luck he obtains that both orange and blue are missing. He then switches to a new model based on a new interpretation of the first row: namely [1 Y] 1R, (and neither blue, nor orange). The third row is played both to try new places for red and to add a new color. Getting 4 pins as a result, the player concludes that the colors of the solution must be yellow, red and green; then, given that there is only one yellow, he considers [1Y] [2R]

[7] The game consists in discovering a hidden row of five colored pawns. One player (the leader) hides a configuration of pawns. The second player can then dispose on a grid a tentative configuration of pawns, and the leader confirms by posting pins (on the right) indicating how many pawn have been discovered. A white pin means a good position and color, and a black one a misplaced color. The rows remain visible during the game, and the player has to find out the solution in a limited number of attempts.

[2G] (which seems more likely than [1Y] [3R] [1G]). He then begins to reason on places and supposes that on the first row, the first yellow pawn on the left is well placed (noted [– – Y – –]). Therefore, the red in the middle of the third row must be misplaced and the two greens must be correctly placed, with one of the two remaining reds as well (see the diagrammatic reasoning in Figure 5). He then concludes that the solution in this case necessarily is [R R Y G G]; but this is found to contradict the answer on row 3, which would then have four white pins.

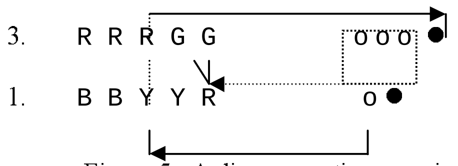

Figure 5: A diagrammatic reasoning

So the player is led to backtrack and reconsider the place of the yellow pawn on row 1: [– – –Y –]. A graphic reasoning very similar to the preceding reveals that the first green pawn is misplaced but that the second one is correct ([– – –Y G], with two of the reds at the beginning). The player then tries a fourth plausible configuration, but is unlucky. Nevertheless, [1Y] [2R] [2G] is confirmed and he knows by experience that getting 3 white pins and 2 black ones means that two pawns have to be exchanged to get the solution. Note that this automatic inference is directly activated by the pattern of pins, and the conclusion directly expressed on the representation. He then makes another attempt [G R R Y G] but is unlucky again. However, a graphic comparison of common parts of rows 3, 4, and 5, leads him to the conclusion that the two framed pawns (R and G) on the right have to be mapped onto either two white pins, or onto a white and a black one (see Figure 6). This strategy is anchored on operations of pattern matching between lines. (The strategy carried out can thus again be called graphic or visual).

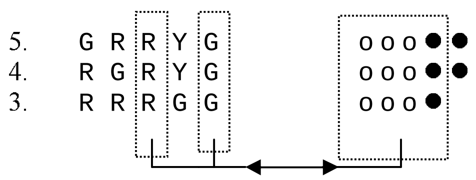

Figure 6: The mapping of common parts

The first hypothesis yields [– – R Y G], which is excluded by the last rows. Since a red can't be in the last position (due to the hypothesis on the first row), the green occupying the last position must be correctly placed, and the red in the middle misplaced, yielding [R R G Y G], which is revealed to be the solution.

Conclusion

The distinction between iconic and linguistic representations has been (partially) analyzed in the literature of the nineties. It has been demonstrated that iconic representations may be taken as syntactical objects in valid inferential systems, sometimes graphically rule-based. Furthermore, in most diagrammatic systems, the consequences of initials facts are included *de facto* in the representation and do not require extra computation. Diagrammatic representations seem to differ mainly from linguistic ones by having a more limited power of abstraction, thereby augmenting computational efficiency. This study also reveals that diagrammatic and logico-linguistic representations have distinct properties, sometimes dual and complementary. This makes their combining in hybrid representation systems (HRS) particularly attractive and promising. The study of such systems opens new research perspectives in the domains of cognitive science and natural language semantics, as well as in complex systems simulation and artificial intelligence in general.

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